ENTROPY DISSIPATION ESTIMATES IN A ZERO–RANGE DYNAMICS

PIETRO CAPUTO AND GUSTAVO POSTA

Abstract. We study the exponential decay of relative entropy functionals for zero–range processes on the complete graph. For the standard model with rates increasing at infinity we prove entropy dissipation estimates, uniformly over the number of particles and the number of vertices.

2000 MSC: 60K35
Key words: Zero–range dynamics, Entropy dissipation, Modified logarithmic Sobolev inequalities.

1. Introduction, Models and Results

Functional estimates such as Poincaré and logarithmic Sobolev inequalities play an important role in the study of approach to stationarity for Markov semigroups, see e.g. [23] for a recent survey. Logarithmic Sobolev inequalities are known to imply exponential decay of relative entropy which in turn provides a natural way to bound mixing times in total variation norm [12]. As we can see already in simple birth–and–death processes, however, in discrete settings logarithmic Sobolev inequalities may become an unnecessarily strong requirement if we are interested in decay to equilibrium in relative entropy or total variation. Motivated by this observation, modified versions of the logarithmic Sobolev inequality have been recently proposed and studied by several authors [1, 11, 13, 14, 2]. As emphasized in [11, 13, 14, 2] a key estimate is the one relating directly the relative entropy functional and its time–derivative along the semigroup. Such entropy dissipation inequalities have been extensively studied in the literature on the approach to equilibrium for the Boltzmann equation, see [25] and references therein. Our aim in this paper is to investigate the validity of entropy dissipation bounds for some models of interacting random walks on the complete graph known as zero–range processes.

The complete graph zero–range dynamics is the continuous time Markov chain described as follows. For each positive integer L we consider the set of vertices $V_L = \{1, \dots, L\}$, the state space is the product $\Omega_L = \mathbb{N}^L$ and a configuration $\eta \in \Omega_L$ is interpreted as an occupation number vector, i.e. η_x is the number of particles at $x \in V_L$. At each vertex $x \in V_L$ we associate a rate function $c_x : \mathbb{N} \to \mathbb{R}$ such that $c_x(0) = 0$ and $c_x(n) > 0$ for every $n \geqslant 1$. We often extend c_x to a function on Ω_L by setting $c_x(\eta) = c_x(\eta_x)$. Every vertex $x \in V_L$ waits an exponentially distributed time with mean $1/c_x$ before one particle is moved from x to a uniformly chosen vertex of V_L. More precisely, the Markov generator is given by

$$\mathcal{L}f = \frac{1}{L}\sum_{x,y} c_x \nabla_{xy} f\,, \tag{1.1}$$

with the sum extending over all $x, y \in V_L$. Here $\nabla_{xy} f$ stands for the gradient $f^{xy} - f$, with $f^{xy}(\eta) = f(\eta^{xy})$, η^{xy} being the configuration in which a particle has been moved from x to y, i.e. $(\eta^{xy})_x = \eta_x - 1$, $(\eta^{xy})_y = \eta_y + 1$, and $(\eta^{xy})_z = \eta_z$, $z \neq x, y$. We agree that

$\eta^{xy} = \eta$, if $\eta_x = 0$. Note that if the functions c_x were all linear, i.e. $c_x(n) = \lambda_x n$, $\lambda_x > 0$, the resulting random walks on the complete graph with L vertices would be independent. The interaction is therefore hidden in the non–linearity of c_x and has zero–range in the sense that jump rates out of x are only determined by the configuration at x. The process is reversible w.r.t. the product measure $\mu_L(\eta) = \prod_{x\in V_L} \mu_x$, where μ_x is the probability on $\mathbb{N}$ given by

$$\mu_x(0) = \frac{1}{Z_x}\,, \quad \mu_x(n) = \frac{1}{Z_x}\prod_{k=1}^{n}\frac{1}{c_x(k)}\,. \tag{1.2}$$

Since the process conserves the initial number of particles, letting $\nu := \nu_{L,N}$ denote the probability μ_L conditioned on the event $N = \sum_{x\in V_L}\eta_x$, we obtain, for every $N \geqslant 1$ and $L \geqslant 2$, an irreducible finite state Markov chain with reversible measure $\nu_{L,N}$. The associated Dirichlet form is given by

$$\mathcal{E}_\nu(f,g) = -\nu\left[f(\mathcal{L}g)\right] = \frac{1}{2L}\sum_{x,y}\nu\left[c_x\nabla_{xy}f\nabla_{xy}g\right]\,, \tag{1.3}$$

where f, g are arbitrary functions and the notation $\nu[f]$ is used for the expectation $\int f\mathrm{d}\nu$. Local variants of the zero–range dynamics have been considered in the literature, especially in connection with hydrodynamical limits [16]. If we allow, for instance, a particle at x to jump to $x+1$ or $x-1$ only, we have the local Dirichlet form

$$\mathcal{D}_\nu(f,g) = \frac{1}{2}\sum_{x=1}^{L-1}\nu\left[c_x\nabla_{x,x+1}f\nabla_{x,x+1}g\right]\,. \tag{1.4}$$

Because of the permutation symmetry of the model it is natural to study the complete graph dynamics, which is more tractable from the analytical point of view. Moreover, it turns out that in some cases sharp estimates on the decay to equilibrium for the local variants are deduced from the corresponding bounds on the complete graph, see e.g. [5, 24].

Let us now recall the notion of entropy and the associated inequalities. As usual the the entropy of a function $f \geqslant 0$ is written $\mathrm{Ent}_\nu(f) = \nu[f\log f] - \nu[f]\log\nu[f]$. When $f \geqslant 0$ and $\nu[f] = 1$, $\mathrm{Ent}_\nu(f)$ coincides with the relative entropy of the probability νf w.r.t. ν. Setting $f_t = e^{t\mathcal{L}}f$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Ent}_\nu(f_t) = -\,\mathcal{E}_\nu(f_t,\log f_t)\,. \tag{1.5}$$

Therefore the entropy dissipation constant

$$\gamma(L,N) = \sup_{f>0}\frac{\mathrm{Ent}_\nu(f)}{\mathcal{E}_\nu(f,\log f)}\,, \tag{1.6}$$

is the best constant γ such that

$$\mathrm{Ent}_\nu(f_t) \leqslant e^{-t/\gamma}\,\mathrm{Ent}_\nu(f)\,, \tag{1.7}$$

for every non–negative function f.

Since (see e.g. [2, 14])

$$\mathcal{E}_\nu(f,\log f) \geqslant 4\,\mathcal{E}_\nu(\sqrt{f},\sqrt{f})\,, \tag{1.8}$$

we see that (1.7) is implied by the usual logarithmic Sobolev inequality. Namely, if $s(L,N)$ denotes the logarithmic Sobolev constant defined by (1.6) with $\mathcal{E}_\nu(f,\log f)$ replaced by $\mathcal{E}_\nu(\sqrt{f},\sqrt{f})$, then $\gamma(L,N) \leqslant s(L,N)/4$. The name "modified" logarithmic Sobolev constant is sometimes used for $\gamma(L,N)$. Note that $s(L,N)$ can be much larger than $\gamma(L,N)$. As an example, consider the simple random walk on the complete graph with L vertices,

which corresponds to the case $N = 1$ with homogeneous rates: $c_x = c_y$, all $x, y \in V_L$. Simple computations show that in this case the logarithmic Sobolev constant $s(L, 1)$ grows with L as $\log L$ while $\gamma(L, 1)$ remains bounded.

Our main result is obtained under the hypothesis of homogeneous Lipschitz rates increasing at infinity. We formulate this as follows.

There exists $c : \mathbb{N} \to \mathbb{R}_+$ such that $c_x(n) = c_y(n) = c(n)$, for all x, y and $n \in \mathbb{N}$. Moreover $c(0) = 0$, $c(k) > 0$ for every $k \geqslant 1$, and there exist $C < \infty$, $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$c(m) - c(n) \geqslant \delta \,, \tag{H.1}$$

for every $n \in \mathbb{N}$ and $m \geqslant n + n_0$, and

$$\sup_{n \geqslant 0} |c(n+1) - c(n)| \leqslant C \,. \tag{H.2}$$

Zero–range processes satisfying (H.1) and (H.2) have been extensively studied [19, 5, 10, 15]. Using a version of the Lu–Yau martingale approach [22], Landim, Sethuraman and Varadhan [19] proved that the spectral gap of the local zero–range dynamics scales diffusively. For the complete graph model a uniform spectral gap estimate was proved in [5] following the Carlen–Carvalho–Loss approach to the determination of the spectral gap [7]. Using a version of the Cancrini–Martinelli duplication method [3, 4] Dai Pra and Posta [9, 10] have recently established a diffusive estimate for the logarithmic Sobolev constant of the local dynamics.

We will show that under (H.1) and (H.2) the entropy dissipation constant $\gamma(L, N)$ has a uniform upper bound.

Theorem 1.1. *Assume (H.1) and (H.2). Then*

$$\sup_{L \geqslant 2} \sup_{N \geqslant 1} \gamma(L, N) \, < \, \infty \,. \tag{1.9}$$

We conclude with some remarks on this result and on the organization of the paper.

1. As in [19, 5, 10] our estimate is uniform over the number of particles N. This uniformity can no longer be expected if one drops the assumption (H.1), see for instance [24] where the spectral gap of the complete graph model with constant rates is shown to be of order $L^2/(N^2 + L^2)$.

2. Following the standard martingale approach, our proof consists in setting up a recursion on the number of vertices L. The first step requires a one–vertex entropy dissipation estimate. This is established in section 2 as a consequence of a more general one–dimensional bound for log–concave measures on $\mathbb{N}$ which is of independent interest. The rest of the proof is given in section 3. Here we need to adapt techniques developed for spectral gap and logarithmic Sobolev inequalities to the more delicate entropy dissipation estimate. In particular, the one–vertex bound is shown to produce certain covariance terms in subsection 3.1. The crucial bound on these covariances is established in subsections 3.2–3.4 by combining the methods of [18] and [10].

3. It is natural to try to extend the result to the case of inhomogeneous rates. One can consider the rates $c_x(n) = \lambda_x c(n)$ with constants $\lambda_x \in [a_1, a_2]$ for some $0 < a_1 < a_2 < \infty$ and $c(\cdot)$ satisfying (H.1) and (H.2). For this model we conjecture that there exists $C < \infty$ depending only on a_1, a_2 and the constants appearing in (H.1), (H.2) such that

$$\gamma(L, N) \leqslant C \,, \tag{1.10}$$

uniformly in N, L. We see no serious difficulty in adapting the one–vertex estimate of section 2 and the arguments up to and including subsection 3.1 to this more general setting. On the other hand, the extension of the covariance estimate to this case seems to be more delicate. In analogy with [17] one can take the $\bar\lambda = \{\lambda_x\}$ as the realization of an i.i.d. random environment. The covariance terms produce new terms involving fluctuations of the environment. Combining the strategy of [17] with our arguments in sesction 3 one could possibly show that the bound (1.10) holds with a constant $C = C(\bar\lambda)$ depending on the random field and such that $C < \infty$ almost surely. Establishing the full conjecture (1.10) however seems to be a more challenging problem which deserves further investigation. If the rates are assumed to be pointwise increasing a first result is the following perturbative bound we obtained in [6] by a discrete version of the Bakry–Emery Γ_2 criterium. Suppose the rates c_x are arbitrary functions satisfying: there exist $\lambda > \delta > 0$ such that

$$\lambda \leqslant c_x(n+1) - c_x(n) \leqslant \lambda + \delta\,, \tag{1.11}$$

for every x and n; then $\gamma(L,N) \leqslant (\lambda-\delta)^{-1}$ for all $L \geqslant 2$ and $N \geqslant 1$. We believe that a uniform estimate on $\gamma(L,N)$ as in (1.9) should hold under (1.11) for any $\delta > 0$ without the restriction $\delta < \lambda$. However, as explained in [6] the Γ_2 approach breaks down when there is no restriction on δ. For the spectral gap the situation is easier. In fact, as shown in [6], assuming (1.11) one has that the spectral gap is bounded below by λ independently of δ.

2. One–vertex estimates

The goal of this section is to show that the one–vertex marginal of the canonical measure ν satisfies a uniform entropy dissipation bound, see Proposition 2.1 below. From now on the rate function $c : \mathbb{N} \to \mathbb{R}$ is assumed to satisfy the conditions (H.1) and (H.2). For any $L \geqslant 2$ and $N \geqslant 1$ we write as usual $\nu = \nu_{L,N}$ for the homogeneous zero–range canonical measure associated to the rate function c. We also write ν_x for the marginal of ν at x, i.e. $\nu_x(n) = \nu(\eta_x = n)$.

Proposition 2.1. *There exists $C < \infty$ such that, for any $L \geqslant 2, N \geqslant 1$, $x \in V_L$ and for any function $u : \mathbb{N} \to \mathbb{R}_+$ with $\nu_x[u] = 1$ we have*

$$\sum_{n=0}^{N} \nu_x(n) u(n) \log u(n) \leqslant C \sum_{n=0}^{N} \nu_x(n) c(n) [u(n) - u(n-1)] \log \frac{u(n)}{u(n-1)}\,. \tag{2.1}$$

The proof will be based on the one–dimensional estimate established in [6], which we recall below.

2.1. A one–dimensional estimate.

Let $\mu : \mathbb{N} \to [0,1]$ be a probability vector and consider the birth and death process with birth rate $r_+(n)$ and death rate $r_-(n)$ satisfying the detailed balance w.r.t. μ:

$$r_-(n)\mu(n) = r_+(n-1)\mu(n-1)\,, \quad n \geqslant 1\,.$$

We assume $r_-(0) = 0$. The following estimate can be found in [6].

Lemma 2.2. *Let r_-, r_+ satisfy*

$$r_-(n+1) - r_-(n) \geqslant \delta_- \tag{2.2}$$

$$r_+(n) - r_+(n+1) \geqslant \delta_+\,, \tag{2.3}$$

with some constants $\delta_-, \delta_+ \geqslant 0$*. Then, for every* $u \geqslant 0$ *such that* $\mu[u] = 1$ *we have*

$$\sum_{n=0}^{\infty} \mu(n)\, u(n)\, \log u(n) \leqslant \delta^{-1} \sum_{n=0}^{\infty} \mu(n)\, r_-(n)\, [u(n) - u(n-1)] \log \frac{u(n)}{u(n-1)}\,, \qquad (2.4)$$

where $\delta := \delta_- + \delta_+$*.*

2.2. **1D–equivalence with the case of increasing rates.** The next step is the following equivalence lemma, whose proof can be found in [6]. Let n_0 be the constant appearing in (H.1). We define

$$\tilde c(k) = c(k) + \frac{1}{n_0} \sum_{j=1}^{n_0-1} \frac{n_0 - j}{n_0} \left[c(k+j) + c(k-j) - 2c(k)\right], \qquad k \geqslant n_0\,. \qquad (2.5)$$

When $k < n_0$ we simply set $\tilde c(k) = \tilde c(n_0) k/n_0$. Let us call $\tilde\mu$ the one–coordinate zero–range measure obtained from $\tilde c$, i.e.

$$\tilde\mu(0) = \frac{1}{\tilde Z}\,, \quad \tilde\mu(n) = \frac{1}{\tilde Z} \prod_{k=1}^{n} \frac{1}{\tilde c(k)}\,. \qquad (2.6)$$

Lemma 2.3. *The rate function* $\tilde c$ *is uniformly increasing: there exists* $\delta > 0$ *such that for every* $k \in \mathbb{N}$

$$\tilde c(k+1) - \tilde c(k) \geqslant \delta\,. \qquad (2.7)$$

Moreover, μ *and* $\tilde\mu$ *are equivalent: there exists* $C \in [1, \infty)$ *such that for every* $n \in \mathbb{N}$

$$\frac{1}{C} \leqslant \frac{\tilde\mu(n)}{\mu(n)} \leqslant C\,. \qquad (2.8)$$

2.3. **Proof of Proposition 2.1.** From a standard comparison result (see e.g. [20], Lemma 1.2) Proposition 2.1 follows if we can prove that ν_x is equivalent to a probability $\hat\nu_x$ on $\mathbb{N}$ for which the estimate (2.1) is known to hold. Here equivalence means a double bound as in (2.8). Since this notion will be used repeatedly in what follows we introduce a special notation for it: We say that $a : \mathbb{N} \to \mathbb{R}_+$ is equivalent to $b : \mathbb{N} \to \mathbb{R}_+$ and write $a \asymp b$ whenever there exists a universal constant $C \in [1, \infty)$ (independent of L and N) such that $C^{-1} \leqslant a/b \leqslant C$.

Recall the notation μ_L for the product $\otimes_{x \in V_L} \mu_x$. We shall use the shortcut notation $\mu_L(k)$ for the probability of the event $\sum_{j=1}^{L} \eta_j = k$, for every $L \geqslant 2$ and $k \in \mathbb{N}$. By definition

$$\nu_x(n) = \mu_x(n)\, \frac{\mu_{L-1}\,(N-n)}{\mu_L\,(N)}\,. \qquad (2.9)$$

Let $\tilde\mu_x$ denote the one–vertex measure with rate $\tilde c$ given by (2.5) and write $\tilde\mu_L = \tilde\mu_x \otimes (\otimes_{y \in V_L \setminus \{x\}} \mu_y)$. From Lemma 2.3 we know that $\mu_x \asymp \tilde\mu_x$ and $\mu_L \asymp \tilde\mu_L$. Therefore $\nu_x \asymp \tilde\nu_x$ where

$$\tilde\nu_x(n) = \tilde\mu_x(n)\, \frac{\mu_{L-1}\,(N-n)}{\tilde\mu_L\,(N)}\,. \qquad (2.10)$$

We will use the following lemma.

Lemma 2.4. *Let* $\hat\nu_x$ *be a probability on* $\{0, 1, \ldots, N\}$ *such that the function*

$$V(n) := -\log \frac{\hat\nu_x(n)}{\tilde\mu_x(n)} \qquad (2.11)$$

satisfies

$$\nabla^2 V(n) = V(n+2) + V(n) - 2V(n+1) \geqslant 0\,, \qquad n = 0,1,\dots,N-2\,. \tag{2.12}$$

Then, for every function $u : \mathbb{N} \to \mathbb{R}_+$ *with* $\hat\nu_x[u] = 1$ *we have*

$$\sum_{n=0}^{N} \hat\nu_x(n) u(n) \log u(n) \leqslant C \sum_{n=1}^{N} \hat\nu_x(n) c(n) [u(n) - u(n-1)] \log \frac{u(n)}{u(n-1)}\,. \tag{2.13}$$

where C *is a constant depending only on the parameters appearing in (H.1) and (H.2).*

Proof. We extend $\hat\nu_x$ to a probability on $\mathbb{N}$ by setting $\hat\nu_x(k) = 0$, $k \geqslant N+1$. We apply Lemma 2.2 with $\mu = \hat\nu_x$, $r_-(n) = \tilde c(n)$. Then, by reversibility and (2.11):

$$r_+(n) = \tilde c(n+1)\, \frac{\hat\nu_x(n+1)}{\hat\nu_x(n)} = e^{-\nabla V(n)}\,.$$

By our log–concavity assumption (2.12) we have $r_+(n) - r_+(n+1) \geqslant 0$, $n = 0,1,\dots,N-1$. Moreover, by Lemma 2.3 $\tilde c(n+1) - \tilde c(n) \geqslant \delta$ for some $\delta > 0$. Therefore, by Lemma 2.2 (with $\delta_+ = 0$) we have the desired estimate (2.13) with $\tilde c$ in place of c, and (2.13) follows from the equivalence $\tilde c \asymp c$. □

Thanks to the equivalence $\nu_x \asymp \tilde\nu_x$ and (2.10), the proof of Proposition 2.1 is an immediate consequence of Lemma 2.4 if we can prove

$$\mu_{L-1}\,(N-n) \asymp e^{-V(n)}\,, \tag{2.14}$$

with a function V satisfying (2.12). To prove (2.14) we introduce the standard grand–canonical zero–range measures. For every $\alpha > 0$ and every vertex x we consider the measures

$$\mu_{x,\alpha}(0) = \frac{1}{Z_\alpha}\,, \quad \mu_{x,\alpha}(n) = \frac{\alpha^n}{Z_\alpha} \prod_{k=1}^{n} \frac{1}{c(k)}\,. \tag{2.15}$$

For every $\rho > 0$, let $\alpha_\rho > 0$ denote the unique value of α such that

$$\sum_{n=0}^{\infty} n \mu_{x,\alpha}(n) = \rho\,. \tag{2.16}$$

It is customary to write simply $\mu_{x,\rho}$ for μ_{x,α_ρ}. Similarly we denote by $\mu_{L,\rho}$ the product $\otimes_{x\in V_L} \mu_{x,\rho}$. Setting $\rho_n := (N-n)/(L-1)$, for every $n \leqslant N-1$ we can write

$$\mu_{L-1}\,(N-n) = (\alpha_{\rho_n})^{n-N} \left(\frac{Z_{\alpha_{\rho_n}}}{Z_1}\right)^{L-1} \mu_{L-1,\rho_n}\,(N-n)\,. \tag{2.17}$$

The idea is to use (2.17) for all values of n except those for which $N-n$ becomes too small. Therefore we fix an integer $m > 0$, set $N_0 = N - m$, and will use the identity (2.17) for all $n \leqslant N_0$. Here we proceed as follows. Denoting by σ_ρ^2 the variance of $\mu_{x,\rho}$ we have the following well known bounds see e.g. [19, 10]:

$$\sigma_\rho^2 \asymp \rho \tag{2.18}$$

$$\mu_{L,\rho}\,(\rho L) \asymp (\sigma_\rho^2 L)^{-\frac12}\,. \tag{2.19}$$

This implies $\mu_{L-1,\rho_n}\,(N-n) \asymp (N-n)^{-\frac12}$. Therefore from (2.17)

$$\mu_{L-1}\,(N-n) \asymp e^{-\tilde V(n)}\,, \tag{2.20}$$

where, for every $t \in [0, N)$ we define $\rho_t = (N-t)/(L-1)$ and

$$\tilde{V}(t) = (N-t)\log\alpha_{\rho_t} - (L-1)\log(Z_{\alpha_{\rho_t}}/Z_1) + \frac{1}{2}\log(N-t)\,. \tag{2.21}$$

We now prove that $\tilde{V}$ is convex if t is not too close to N, i.e. $\tilde{V}''(t) \geqslant 0$, $t \leqslant N_0$. Clearly $\tilde{V}''(t) = \varphi''(t) - (2(N-t)^2)^{-1}$ with $\varphi(t) := (N-t)\log\alpha_{\rho_t} - (L-1)\log(Z_{\alpha_{\rho_t}}/Z_1)$. We have

$$\varphi'(t) = -\log(\alpha_{\rho_t}) + (N-t)\frac{\mathrm{d}}{\mathrm{d}t}\log(\alpha_{\rho_t}) - (L-1)\frac{\mathrm{d}}{\mathrm{d}t}\log(Z_{\alpha_{\rho_t}})\,.$$

Using (2.16) we see that $\frac{\mathrm{d}}{\mathrm{d}t}\log(Z_{\alpha_{\rho_t}}) = \rho_t\frac{\mathrm{d}}{\mathrm{d}t}\log(\alpha_{\rho_t})$ and the last two terms in the expression for $\varphi'(t)$ cancel each other. We then have $\varphi''(t) = -\frac{\mathrm{d}}{\mathrm{d}t}\log(\alpha_{\rho_t})$. Reasoning as above and using $\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = -1/(L-1)$ we have $\varphi''(t) = 1/(L-1)\sigma^2_{\rho_t}$. Therefore, for some independent $C \in [1,\infty)$

$$\tilde{V}''(t) = \frac{1}{(L-1)\sigma^2_{\rho_t}} - \frac{1}{2(N-t)^2} \geqslant \frac{1}{C(N-t)} - \frac{1}{2(N-t)^2}\,, \tag{2.22}$$

where in the last estimate we have used (2.18). Then $\tilde{V}''(t) \geqslant 0$ for all $N - t \geqslant C/2$. This implies – by integration – that $\nabla\tilde{V}(n) \leqslant \nabla\tilde{V}(n+1)$ at least for all $n \leqslant N - 2 - C/2$. Setting e.g. $m = [C]$ we have shown that $\nabla^2\tilde{V}(n) \geqslant 0$, $n \leqslant N_0 - 2 = N - m - 2$.

We still have to deal with the case $N - n \leqslant m$. Here we use the fact that

$$\mu_{L-1}(k) \asymp L^k \mu_x(0)^L\,, \qquad k \leqslant m\,. \tag{2.23}$$

To prove the lower bound in (2.23) we simply observe that putting k particles in k different sites one has, for some m–dependent $C < \infty$

$$\mu_{L-1}(k) \geqslant \binom{L-1}{k}\mu_x(1)^k\mu_x(0)^{L-k} \geqslant \frac{1}{C}\,L^k\mu_x(0)^L\,, \qquad k \leqslant m\,.$$

Similarly the upper bound is obtained by requiring at least $L-1-k$ sites to be empty:

$$\mu_{L-1}(k) \leqslant \sum_{\ell\,\geqslant\, L-1-k}\binom{L-1}{\ell}\mu_x(0)^\ell \leqslant C\,L^k\mu_x(0)^L\,, \qquad k \leqslant m\,.$$

Summarizing, from (2.20) and (2.23) we have obtained that, for every fixed $K \in (0,\infty)$ the equivalence (2.14) holds with the function $V = V_K$ given by

$$V(n) = \begin{cases} \tilde{V}(n) & n \leqslant N_0 \\ (n-N)\log L - L\log\mu_x(0) + K^{n-N_0} & N_0 < n \leqslant N \end{cases} \tag{2.24}$$

Note that the addition of the term K^{n-N_0} in (2.24) does not break the equivalence since $n - N_0 \leqslant m$. What we have seen in (2.22) implies $\nabla^2 V(n) \geqslant 0$ for $n \in [0, N_0 - 2]$. We are left with the case $n \geqslant N_0 - 1$. But this is easily obtained by taking the constant K sufficiently large. For instance: from (2.23) we know that, for some universal constant $C < \infty$ $\tilde{V}(N_0) \geqslant (N_0 - N)\log L - L\log\mu_x(0) - C$, so that $\nabla V(N_0) \leqslant \log L + K + C$. On the other hand $\nabla V(N_0+1) = \log L + K^2 - K$. For K large this gives $\nabla^2 V(N_0) \geqslant 0$. Similar reasoning applies for the remaining values of $n \geqslant N_0 - 1$. This ends the proof of the claim in (2.14) and concludes the proof of Proposition 2.1.

3. Proof of Theorem 1.1

The proof of Theorem 1.1 is based on a variant of the martingale recursive method developed in [22], see also [19, 18, 13, 21, 26, 8]. We set

$$\gamma(L) = \sup_{N \geqslant 1} \gamma(L,N)\,. \tag{3.1}$$

Note that the result of [10] on the logarithmic Sobolev inequality for the local dynamics defined by (1.4) implies that $\gamma(L) < \infty$ for every L. We are going to prove

$$\sup_L \gamma(L) < \infty\,. \tag{3.2}$$

To this end we start with the usual decomposition of entropy and write, for $f > 0$

$$\mathrm{Ent}_\nu(f) = \frac{1}{L}\sum_x \nu\left[\mathrm{Ent}_\nu(f \,|\, \eta_x)\right] + \frac{1}{L}\sum_x \mathrm{Ent}_\nu(f_x)\,, \tag{3.3}$$

where $\mathrm{Ent}_\nu(f \,|\, \eta_x)$ denotes the entropy of f w.r.t. $\nu[\cdot \,|\, \eta_x]$ (the measure ν conditioned to have a given number of particles η_x at x) and we have defined

$$f_x(\eta) = f_x(\eta_x) = \nu[f \,|\, \eta_x]\,.$$

Since, for every given $0 \leqslant \eta_x \leqslant N$, the measure $\nu[\cdot \,|\, \eta_x]$ coincides with the canonical zero–range measure on $L-1$ vertices with total particle number $N - \eta_x$, we can estimate, for every η_x

$$\mathrm{Ent}_\nu(f \,|\, \eta_x) \leqslant \gamma(L-1)\frac{1}{L-1}\sum_{y\neq x}\sum_{z \neq x} \nu\left[c_y\, \nabla_{yz} f\, \nabla_{yz}\log f \,|\, \eta_x\right]\,.$$

Taking ν–expectation and averaging the above expression over x we obtain that (3.3) is bounded above by

$$\gamma(L-1)\,\frac{L-2}{L-1}\,\mathcal{E}_\nu(f, \log f) + \frac{1}{L}\sum_x \mathrm{Ent}_\nu(f_x)\,. \tag{3.4}$$

The next two subsections will explain how to estimate the second term in (3.4). Here we anticipate that the final result (see (3.21) and (3.25) below) will be that for every $\epsilon > 0$ there exist two constants $\ell_\epsilon, C_\epsilon < \infty$ independent of L and N such that for all $L \geqslant \ell_\epsilon$ we have

$$\sum_x \mathrm{Ent}_\nu(f_x) \leqslant \epsilon\, \mathrm{Ent}_\nu(f) + C_\epsilon\, \mathcal{E}_\nu(f, \log f)\,. \tag{3.5}$$

Once the above result is available it is easy to end the proof of (3.2). Indeed, from (3.5) and (3.4) we obtain

$$\left(1 - \frac{\epsilon}{L}\right)\gamma(L) \;\leqslant\; \frac{L-2}{L-1}\,\gamma(L-1) + \frac{C_\epsilon}{L}\,, \qquad L \geqslant \ell_\epsilon\,, \tag{3.6}$$

which implies the claim (3.2) if ϵ is sufficiently small (e.g. $\epsilon < \frac{1}{2}$).

3.1. From one–vertex estimate to covariances.

Let us recall the following change of variable relation: for any function f and any pair of vertices x, y

$$\nu[c_x f] = \nu[c_y f^{yx}]\,. \tag{3.7}$$

The above is an immediate consequence of the definitions of the symbols involved and the fact that, for any $\eta \in \Omega_L$ with $\eta_x \geqslant 1$ we have

$$\frac{\nu(\eta^{xy})}{\nu(\eta)} = \frac{\mu_x(\eta_x - 1)\mu_y(\eta_y+1)}{\mu_x(\eta_x)\mu_y(\eta_y)} = \frac{c_x(\eta_x)}{c_y(\eta_y+1)}\,.$$

We start our proof of the claim (3.5) with an application of Proposition 2.1 to the function $u = f_x/\nu[f_x]$. Here and in the rest of this subsection x is an arbitrary fixed vertex. We have

$$\mathrm{Ent}_\nu(f_x) \leqslant C \sum_{n=0}^{N} \nu_x(n)c(n)[f_x(n) - f_x(n-1)] \log \frac{f_x(n)}{f_x(n-1)} \,. \tag{3.8}$$

To estimate the R.H.S. of (3.8) we first rewrite things as follows. For every vertex $y \neq x$ and for every n we define the functions

$$g_{x,y,n}(\eta) = \frac{c_y(\eta)}{\nu[c_y \,|\, \eta_x = n]} \,, \quad g_{x,n}(\eta) = \frac{1}{L-1} \sum_{y \neq x} g_{x,y,n}(\eta) \,. \tag{3.9}$$

In order to simplify notations, below we will write $\nu[\cdot \,|\, n]$ for $\nu[\cdot \,|\, \eta_x = n]$. Formula (3.7) can be used to deduce the identity

$$f_x(n) = \nu[f \,|\, n] = \nu[g_{x,y,n-1} f^{yx} \,|\, n-1] \,, \tag{3.10}$$

valid for every $y \neq x$ and $n \geqslant 1$. Indeed, write $\chi_{x,n}(\eta)$ for the indicator function of the event $\{\eta_x = n\}$. Then $(\chi_{x,n})^{yx} = \chi_{x,n-1}$ and

$$\nu[f\chi_{x,n}] = \frac{1}{c(n)}\, \nu[c_x f \chi_{x,n}] = \frac{1}{c(n)}\, \nu[c_y f^{yx} \chi_{x,n-1}] \,.$$

When $f = 1$ this shows that $\nu[c_y \,|\, n-1] = \frac{c(n)\nu_x(n)}{\nu_x(n-1)}$ and (3.10) follows.

In particular, (3.10) shows that

$$\nu[f \,|\, n] - \nu[g_{x,n-1} f \,|\, n-1] = \frac{1}{L-1} \sum_{y \neq x} \nu\left[g_{x,y,n-1} \nabla_{yx} f \,|\, n-1\right] \,. \tag{3.11}$$

Our first step in the estimate of (3.8) is the next lemma. We recall the standard notation $\mu[f,g] = \mu[fg] - \mu[f]\mu[g]$ for the covariance of two functions f, g w.r.t. a measure μ.

Lemma 3.1. *There exists $C < \infty$ such that for every $f > 0$, $L \geqslant 2$, and $N \geqslant n \geqslant 1$*

$$[f_x(n) - f_x(n-1)] \log \frac{f_x(n)}{f_x(n-1)} \leqslant C \left\{ A_x(n) \,+\, B_x(n) \right\} , \tag{3.12}$$

where we define

$$A_x(n) = (\nu[f \,|\, n] - \nu[g_{x,n-1} f \,|\, n-1]) \,\log \frac{\nu[f \,|\, n]}{\nu[g_{x,n-1} f \,|\, n-1]} \,, \tag{3.13}$$

$$B_x(n) = \frac{\nu[g_{x,n-1}, f \,|\, n-1]^2}{f_x(n) \vee f_x(n-1)} \,. \tag{3.14}$$

Proof. Set $a = f_x(n)$, $b = \nu[g_{x,n-1} f \,|\, n-1]$ and $c = f_x(n-1)$. With the notation $\alpha(a,b) = (a-b)\log(a/b)$, the desired estimate (3.12) can be written as

$$\alpha(a,c) \leqslant C\,\alpha(a,b) + C\,\frac{(b-c)^2}{a \vee c} \,. \tag{3.15}$$

Note that the above inequality cannot hold for all $a, b, c > 0$ without restrictions (take e.g. $c = 1$, $a = b$ and let $b \nearrow \infty$). The point is that in our setting we have $1/C \leqslant b/c \leqslant C$, for some possibly different $C \in [1, \infty)$. To see this recall that $\nu[\eta_y \,|\, n] = (N-n)/(L-1)$ for all n and $y \neq x$ and use $c(n) \asymp n$ to obtain

$$g_{x,n-1}(\eta) \asymp \frac{\sum_{y \neq x} \eta_y}{N - (n-1)} = 1 \,, \quad \nu[\cdot \,|\, n-1] \,-\, a.s. \tag{3.16}$$

for every $n \geqslant 1$. Therefore $b \asymp c$. We now write

$$\alpha(a,c) = c(a/c-1)\log(a/c) = c\,h(t)\,,$$
$$h(t) := t(e^t-1)\,, \quad t := \log(a/c)\,.$$

It is not difficult to check the function h satisfies: for every $C < \infty$

$$\sup_{u \leqslant C}\ \sup_{t \leqslant 2C} \frac{h(t)}{h(t-u)+u^2} < \infty\,, \tag{3.17}$$

$$\sup_{u \leqslant C}\ \sup_{t \geqslant 2C} \frac{h(t)}{h(t-u)} < \infty\,. \tag{3.18}$$

In the rest of this proof we use $C_1, C_2, \dots$ to denote finite positive constants (independent of n, N, L). Setting $u := \log(b/c)$, we know that $u \leqslant C_1$. Suppose first that $a/c \leqslant 2C_1$. Then by (3.17) we know that there exists $C_2 < \infty$ such that $h(t) \leqslant C_2(h(t-u)+u^2)$, i.e.

$$\alpha(a,c) \leqslant C_2\left[c(a/b-1)\log(a/b) + c(\log(b/c))^2\right]\,.$$

The first term above is $c/b\,\alpha(a,b) \leqslant C\alpha(a,b)$. For the second term we use the elementary fact that for every $\delta > 0$, there is $C = C(\delta) < \infty$ such that $|\log(1+x)| \leqslant C\,|x|$, for any $x \geqslant \delta - 1$. With $x = b/c - 1$, this says that the second term is bounded by

$$C_3\,\frac{(b-c)^2}{c} \leqslant C_4\,\frac{(b-c)^2}{a \vee c}\,,$$

where we used the assumption $a/c \leqslant 2C_1$. This completes the proof of (3.15) under this assumption. If $a/c > 2C_1$ we have by (3.18) $h(t) \leqslant C_5 h(t-u)$, i.e.

$$\alpha(a,c) \leqslant C_5\,c\,(a/b-1)\log(a/b) \leqslant C_6\alpha(a,b)\,,$$

which clearly implies (3.15). □

When we insert the estimate of Lemma 3.1 in (3.8) we therefore obtain two terms, corresponding to $A_x(n)$ and $B_x(n)$, respectively. We explain here how to bound the first term. This is a modification of a rather standard convexity argument, see e.g. [13]. The more delicate estimate of the term coming from $B_x(n)$ is given in the next subsection.

Thanks to the identity (3.10) and the convexity of $(a,b) \to (a-b)\log(a/b)$ on $\mathbb{R}_+ \times \mathbb{R}_+$, Jensen's inequality implies

$$A_x(n) \leqslant \frac{1}{L-1}\sum_{y \neq x} \nu\left[g_{x,y,n-1}\nabla_{yx} f\, \nabla_{yx}\log f \;|\, n-1\right]\,. \tag{3.19}$$

Going back to (3.8) and using (see (3.10))

$$\nu_x(n)c(n)g_{x,y,n-1} = \nu_x(n-1)\,c_y\,, \quad y \neq x\,,$$

we see that

$$\sum_{n=0}^{N} \nu_x(n)c(n)A_x(n) \leqslant \frac{1}{L-1}\sum_{y \neq x} \nu\left[c_y\,\nabla_{yx} f\,\nabla_{yx}\log f\right]\,. \tag{3.20}$$

When we sum over x in (3.8), from Lemma 3.1 and (3.20) we obtain

$$\sum_x \mathrm{Ent}_\nu(f_x) \leqslant C\,\mathcal{E}_\nu(f,\log f) + C\sum_x \sum_{n=0}^{N} \nu_x(n)c(n)B_x(n)\,. \tag{3.21}$$

3.2. **The covariance estimate.** We need the following key estimate on covariances.

Proposition 3.2. *Assume (H.1) and (H.2). For every $\epsilon > 0$, there exist finite constants C_ϵ and ℓ_ϵ such that for every $L \geqslant \ell_\epsilon$, $N \geqslant 1$ and for every $f > 0$*

$$\nu\left[f, \sum_x c_x\right]^2 \leqslant N\,\nu[f]\,\Big[C_\epsilon \mathcal{E}_\nu(\sqrt{f},\sqrt{f}) + \epsilon\,\mathrm{Ent}_\nu(f)\Big]\,. \tag{3.22}$$

Before going to the proof we want to make sure this result is indeed sufficient for our claim (3.5) to hold. To this end we fix a vertex x and apply (3.22) by replacing ν with $\nu[\cdot\,|\,n-1] = \nu[\cdot\,|\,\eta_x = n-1]$, L by $L-1$ and N by $N-n+1$. Using the equivalence

$$\nu[c_y\,|\,n-1] \asymp \frac{N-n+1}{L-1}\,, \tag{3.23}$$

we then see that for some $C < \infty$

$$\nu[f, g_{x,n-1}\,|\,n-1]^2 \leqslant \frac{C\,f_x(n-1)}{N-n+1}\Big[C_\epsilon \mathcal{E}_\nu(\sqrt{f},\sqrt{f}\,|\,n-1) + \epsilon\,\mathrm{Ent}_\nu(f\,|\,n-1)\Big]\,. \tag{3.24}$$

Using again (3.23) and the identity $\nu_x(n)c(n) = \nu[c_y\,|\,n-1]\nu_x(n-1)$ we get, with a possibly different constant C

$$\sum_{n=0}^N \nu_x(n)c(n)B_x(n) \leqslant \frac{C}{L}\,\Big[C_\epsilon \mathcal{E}_\nu(\sqrt{f},\sqrt{f}) + \epsilon\,\mathrm{Ent}_\nu(f)\Big]\,, \tag{3.25}$$

where we have used the easily verified estimates

$$\nu[\mathcal{E}_\nu(\sqrt{f},\sqrt{f}\,|\,\eta_x)] \leqslant \mathcal{E}_\nu(\sqrt{f},\sqrt{f})\,,\quad \nu[\mathrm{Ent}_\nu(f\,|\,\eta_x)] \leqslant \mathrm{Ent}_\nu(f)\,.$$

Finally, the desired estimate (3.5) follows from (3.25), (3.21) and the elementary bound (1.7).

We turn to the proof of Proposition 3.2. Let us first recall the covariance estimate proved in [10]. Corollary 3.11 there states that assuming (H.1) and (H.2) one has

$$\nu\left[f, \sum_x c_x\right]^2 \leqslant C\,N\,\nu[f]\,\Big[\nu[f] + C_\epsilon\,L^2\,\mathcal{D}_\nu(\sqrt{f},\sqrt{f}) + \epsilon \mathrm{Ent}_\nu(f)\Big]\,. \tag{3.26}$$

Here C is a finite constant depending only on the parameters appearing in (H.1) and (H.2) and $\mathcal{D}_\nu$ stands for the local Dirichlet form defined in (1.4). The constants ϵ and C_ϵ have the same meaning as in our Proposition 3.2 above. To prove our bound in (3.22) we therefore have to improve the latter result in two ways: first, we need to replace $L^2\,\mathcal{D}_\nu(\sqrt{f},\sqrt{f})$ by $\mathcal{E}_\nu(\sqrt{f},\sqrt{f})$ and second, we have to remove the extra term $\nu[f]$ appearing in (3.26). It turns out that the first improvement requires only straightforward modifications of the argument of [10]. The second, on the other hand, will require some additional work, which will be based on a combination of ideas from [10] and [18]. As in [10] we consider separately the case of small density and the case of densities uniformly bounded away from zero. In the rest of the proof of Proposition 3.2 we adopt the convention that C represents a generic finite constant which may only depend on the parameters appearing in (H.1) and (H.2). When constants depend on a further parameter as e.g. ϵ, M or K we write this explicitly as C_ϵ, C_M or C_K respectively. In all cases it is understood that these constants are independent of L and N. We warn the reader that the numerical value of these constants may change from line to line.

3.3. **Small density.** Here we assume that $\rho := N/L$ satisfies $\rho \leqslant \rho_0$ with ρ_0 a parameter to be taken sufficiently small depending on ϵ. Recall the definition (2.16) of the parameter α_ρ. We use the notations

$$\begin{aligned} \varphi_x(\eta_x) &= c(\eta_x) - \frac{\alpha_\rho}{\rho}\,\eta_x \\ \bar\varphi_x = \varphi_x - \nu[\varphi_x]\,, &\quad \Phi(\eta) = \sum_x \bar\varphi_x\,. \end{aligned} \tag{3.27}$$

Lemma 3.3. *For every $M>0$, there exists $C_M<\infty$ such that*

$$\frac{1}{t}\log\nu\left[\exp t|\Phi|\,\right] \leqslant C_M N\sqrt{\rho}\,t\,, \qquad t\in[0,M]\,. \tag{3.28}$$

Before giving a proof we show that Lemma 3.3 implies that for every $M>0$, there exists $C_M<\infty$ such that for any $f>0$, with $\nu[f]=1$:

$$\nu\left[f,\sum_x c_x\right]^2 \leqslant C\,N\,\left[(C_M\rho)\vee\frac{1}{M}\right]\mathrm{Ent}_\nu(f)\,. \tag{3.29}$$

Of course, by taking ρ_0 small enough, (3.29) gives the desired result (3.22) for small density. To prove (3.29) we use the entropy inequality to write, for every $t>0$

$$\nu\left[f,\sum_x c_x\right] = \nu\left[f\Phi\right] \leqslant \frac{1}{t}\,\log\nu\left[\exp t\Phi\,\right] + \frac{1}{t}\,\mathrm{Ent}_\nu(f)\,.$$

We may apply the above inequality with $-\Phi$ replacing Φ. Therefore, passing to absolute values, Lemma 3.28 gives

$$\left|\nu\left[f,\sum_x c_x\right]\right| \leqslant C_M N\sqrt{\rho}\,t + \frac{1}{t}\,\mathrm{Ent}_\nu(f)\,, \qquad t\in[0,M]\,. \tag{3.30}$$

Set now $\bar t = \sqrt{\frac{M}{N}\,\mathrm{Ent}_\nu(f)}$. If $\bar t\leqslant M$, (3.29) follows immediately by plugging $t=\bar t$ in (3.30). If, however, $\bar t\geqslant M$ we may use the rough bound $|\sum_x c_x|\leqslant C\,N$ to estimate

$$\nu\left[f,\sum_x c_x\right]^2 \leqslant C\,N^2 \leqslant CN\,\frac{1}{M}\,\mathrm{Ent}_\nu(f)\,.$$

We now turn to the proof of Lemma 3.3. Since all our estimates below are easily seen to hold with Φ replaced by $-\Phi$ we may restrict to estimate $\nu\left[\exp t\Phi\right]$ instead of $\nu\left[\exp t|\Phi|\right]$. We consider two different cases: $M\geqslant t\geqslant M\wedge\sqrt{L}/N$ and $t\leqslant M\wedge\sqrt{L}/N$.

Case $M\geqslant t\geqslant M\wedge(\sqrt{L}/N)$. We recall the following bound derived in [10], see (4.80) and (4.88) there:

$$\frac{1}{t}\log\nu\left[\exp t\Phi\,\right] \leqslant \frac{C}{t} + C\sqrt{N} + C_M N\,\rho\,t\,, \qquad t\in[0,M]. \tag{3.31}$$

If $t\geqslant\sqrt{L}/N$ we have $1/t\leqslant N\rho\,t$ and $\sqrt{N}\leqslant N\sqrt{\rho}\,t$. Therefore (3.28) is contained in (3.31) in this case.

Case $t\leqslant M\wedge(\sqrt{L}/N)$. The bound (3.31) is not optimal for small values of t and we need a different approach here. We may proceed as in [18], Lemma 6.5. Without loss of

generality, we assume that L is even. We call $V_{L/2}$ the set of vertices $\{1, 2, \dots, L/2\}$. By Schwarz inequality we have

$$\log \nu \left[\exp t\Phi\right] \leqslant \log \nu \left[\exp 2t\widetilde{\Phi}\right] , \quad \widetilde{\Phi} := \sum_{x \in V_{L/2}} \bar{\varphi}_x \,.$$

For every function g such that $\nu[g] = 0$ we may estimate

$$\nu[e^g] \leqslant \exp\left\{\frac{1}{2}\,\nu\left[g^2 e^{|g|}\right]\right\} . \tag{3.32}$$

This estimate follows from $e^a \leqslant 1 + a + \frac{1}{2}a^2 e^{|a|}$, and $1 + x \leqslant e^x$. We apply this bound to $g = 2t\widetilde{\Phi}$. Using the equivalence of ensembles bound (see e.g. Proposition 4.1 in [10]) we have $\nu[\widetilde{\Phi}^2 \exp 2t|\widetilde{\Phi}|] \leqslant C\mu_{L,\rho}[\widetilde{\Phi}^2 \exp 2t|\widetilde{\Phi}|]$ and therefore

$$\nu\left[\exp 2t\widetilde{\Phi}\right] \leqslant \exp\left\{C\,t^2\,\mu_{L,\rho}\left[\widetilde{\Phi}^2 e^{2t|\widetilde{\Phi}|}\right]\right\} . \tag{3.33}$$

All the estimates below can be obtained for $-\widetilde{\Phi}$ as well as for $\widetilde{\Phi}$ without any change, therefore we will restrict to bound the expression

$$\mu_{L,\rho}\left[\widetilde{\Phi}^2 e^{2t\widetilde{\Phi}}\right] = \sum_{x,y \in V_{L/2}} \mu_{L,\rho}\left[\bar{\varphi}_x \bar{\varphi}_y e^{2t\widetilde{\Phi}}\right] = E_1 + E_2 \,, \tag{3.34}$$

where, using the product structure of $\mu_{L,\rho}$ and writing $\mu_\rho = \mu_{1,\rho}$

$$E_1 := \frac{L}{2}\,\mu_\rho\left[\bar{\varphi}_1^2 e^{2t\bar{\varphi}_1}\right]\mu_\rho\left[e^{2t\bar{\varphi}_1}\right]^{\frac{L}{2}-1} ,$$

$$E_2 := \frac{L}{2}\left(\frac{L}{2} - 1\right)\,\mu_\rho\left[\bar{\varphi}_1 e^{2t\bar{\varphi}_1}\right]^2 \mu_\rho\left[e^{2t\bar{\varphi}_1}\right]^{\frac{L}{2}-2} .$$

Recalling (see e.g.Corollary 6.4 in [19]) that $|\bar{\varphi}_1 - \varphi_1| \leqslant C\frac{\sqrt{1+\rho}}{L}$, we estimate

$$\mu_\rho\left[e^{2t\bar{\varphi}_1}\right] \leqslant e^{t\frac{C}{L}}\,\mu_\rho\left[e^{2t\varphi_1}\right] .$$

From (4.88) in [10], $\mu_\rho\left[e^{2t\varphi_1}\right] \leqslant e^{C_M\rho^2 t^2}$, $t \leqslant M$. Therefore

$$\mu_\rho\left[e^{2t\bar{\varphi}_1}\right]^{\frac{L}{2}-1} \leqslant C_M\,e^{C_M\rho^2 t^2 L} \leqslant C_M \,,$$

the last bound following from $t^2 \leqslant L/N^2$. This gives $E_1 \leqslant C_M L\,\mu_\rho\left[\bar{\varphi}_1^2 e^{2t\bar{\varphi}_1}\right]$. Replacing as above $\bar{\varphi}_1$ with φ_1 we have

$$\mu_\rho\left[\bar{\varphi}_1^2 e^{2t\bar{\varphi}_1}\right] \leqslant \frac{C}{L^2} + C\,\mu_\rho\left[\varphi_1^2 e^{2t\varphi_1}\right] . \tag{3.35}$$

By direct computation (or reasoning as in (4.82),(4.84) and (4.86) in [10]) it is not hard to obtain the bound

$$\mu_\rho\left[\varphi_1^2 e^{2t\varphi_1}\right] \leqslant C_M\rho^2 , \quad t \leqslant M . \tag{3.36}$$

From (3.35) and (3.36), using $\rho \geqslant 1/L$, we have obtained $E_1 \leqslant C_M N\,\rho$. We now look for a similar bound on E_2. We first observe that for any $a \in \mathbb{R}$ we have $ae^a \leqslant a + a^2 e^{|a|}$. Setting $a = 2t\bar{\varphi}_1$ we obtain

$$\mu_\rho\left[\bar{\varphi}_1 e^{2t\bar{\varphi}_1}\right] \leqslant \mu_\rho\left[\bar{\varphi}_1\right] + 2t\mu_\rho\left[\bar{\varphi}_1^2 e^{2t|\bar{\varphi}_1|}\right] .$$

Estimating as in (3.35) and (3.36) once for $\bar\varphi_1$ and once for $-\bar\varphi_1$, the second term above is bounded by $C_M t\,\rho^2$. Since $\mu_\rho[\varphi_1]=0$, direct computations show that $|\mu_\rho[\bar\varphi_1]|\leqslant C(\rho^2\wedge\frac{1}{L})$. Therefore

$$\mu_\rho\left[\bar\varphi_1 e^{2t\bar\varphi_1}\right]\leqslant C\frac{\rho}{\sqrt{L}}+C_M t\,\rho^2\,.$$

Reasoning as above it is not hard to check that the last estimate holds for $-\mu_\rho\left[\bar\varphi_1 e^{2t\bar\varphi_1}\right]$ as well. We then obtain

$$\mu_\rho\left[\bar\varphi_1 e^{2t\bar\varphi_1}\right]^2\leqslant C\frac{\rho^2}{L}+C_M t^2\,\rho^4\,.$$

This implies the estimate $E_2\leqslant C\,L\,\rho^2+C_M L^2\,t^2\,\rho^4$. Using the constraint $t^2\leqslant L/N^2$ this becomes $E_2\leqslant C_M N\,\rho$. In conclusion: from (3.33) and (3.34) we have

$$\frac{1}{t}\log\nu\left[\,\exp t\Phi\,\right]\leqslant C_M N\,\rho\,t\,. \tag{3.37}$$

This ends the proof of Lemma 3.3.

3.4. **Density bounded away from zero.** To prove Proposition 3.2 in the regime $\rho\geqslant\rho_0$ we need the following standard coarse graining procedure. We fix a parameter $K>0$ to be taken sufficiently large in the sequel. Without loss of generality we will assume that K divides L so that the set of vertices V_L is the disjoint union of $\ell:=L/K$ sets of vertices $B_1,\dots,B_\ell$, each of cardinality K. We write $N_j=N_j(\eta)=\sum_{x\in B_j}\eta_x$ for the number of particles in the block B_j and write $\mathcal{G}$ for the σ–algebra generated by the functions $\eta\to N_j(\eta)$, $j=1,\dots,\ell$. In this way, the conditional expectation $\nu[\cdot\,|\,\mathcal{G}]$ becomes the product $\prod_{j=1}^{\ell}\nu_{j,N_j}[\cdot]$, where ν_{j,N_j} denotes the canonical zero–range measure on the j–th block with N_j particles. We start with the decomposition

$$\nu\left[f,\sum_x c_x\right]=\nu\left[\nu\Big[f,\sum_x c_x\,|\,\mathcal{G}\Big]\right]+\nu\left[f,\sum_{j=1}^{\ell}\nu_{j,N_j}\Big[\sum_{x\in B_j}c_x\Big]\right]\,. \tag{3.38}$$

As in [10], Corollary 3.9, it is not hard to prove

$$\nu\left[\nu\Big[f,\sum_x c_x\,|\,\mathcal{G}\Big]\right]\leqslant C_K N\,\nu[f]\,\mathcal{E}_\nu(\sqrt{f},\sqrt{f})\,. \tag{3.39}$$

We now concentrate on a bound on the second term in (3.38). To this end we introduce the following notations. For every x we set $\bar c_x(\eta)=c(\eta_x)-\alpha'_\rho\eta_x$, where $\alpha'_\rho=\frac{\mathrm{d}}{\mathrm{d}\rho}\alpha_\rho$, and, with $\rho_j:=N_j/K$, for ever $x\in B_j$

$$\begin{aligned}G(\rho_j)&=\nu_{j,N_j}[\bar c_x]-\mu_{x,\rho}[\bar c_x]\,,\\ \bar G(\rho_j)&=\nu_{j,N_j}[\bar c_x]-\nu[\bar c_x]\,.\end{aligned}$$

Note that these definition do not depend on the chosen $x\in B_j$. Moreover, $\mu[G(\rho_j)]=0$ and $\nu[\bar G(\rho_j)]=0$. We also set

$$\Psi(\eta)=K\sum_{j=1}^{\ell}\bar G(\rho_j)\,,$$

so that the second term in (3.38) becomes $\nu[f\Psi]$. Therefore our ultimate claim now becomes: for every $\rho_0>0$, for every $\epsilon>0$ there exist constants $K_\epsilon,\ell_\epsilon,C_\epsilon<\infty$ such that

for all $K \geqslant K_\epsilon$ $\ell \geqslant \ell_\epsilon$

$$\nu[f\Psi]^2 \leqslant C\,N\,\nu[f]\,\left(C_\epsilon \mathcal{E}_\nu(\sqrt{f},\sqrt{f}) + \epsilon\,\mathrm{Ent}_\nu(f)\right)\,. \tag{3.40}$$

As a simplifying rule we do not write explicitly the ρ_0–dependence of the various constants.

Lemma 3.4. *For every $M > 0$ there exists $C_M < \infty$ such that*

$$\frac{1}{t}\log\nu\left[\exp t|\Psi|\,\right] \leqslant \frac{C_M N}{\sqrt{K}}\,t\,, \qquad 0 \leqslant t \leqslant \frac{M}{K\sqrt{\rho}}\,. \tag{3.41}$$

Before giving the proof of the lemma we want to show that the estimate (3.41) is sufficient to prove (3.40). As in (3.30), assuming $\nu[f] = 1$, (3.41) allows to estimate

$$|\nu\left[f\Psi\right]| \leqslant \frac{C_M N}{\sqrt{K}}\,t + \frac{1}{t}\,\mathrm{Ent}_\nu(f)\,, \qquad 0 \leqslant t \leqslant \frac{M}{K\sqrt{\rho}}\,. \tag{3.42}$$

Set again $\bar t = \sqrt{\frac{M}{N}\,\mathrm{Ent}_\nu(f)}$. If $\bar t \leqslant \frac{M}{K\sqrt{\rho}}$, plugging $t = \bar t$ in (3.42) we have

$$\nu[f\Psi]^2 \leqslant C\,N\,\left(\frac{C_M}{K} \vee \frac{1}{M}\right)\,\mathrm{Ent}_\nu(f)\,. \tag{3.43}$$

Taking M and K sufficiently large in a suitable way this clearly implies (3.40). The case $\bar t \geqslant \frac{M}{K\sqrt{\rho}}$ is much more delicate. By repeating exactly the computations in [10], see (4.76) there, in this case one arrives at the desired estimate (3.40) except that $\mathcal{E}_\nu(\sqrt{f},\sqrt{f})$ is replaced by $L^2\mathcal{D}_\nu(\sqrt{f},\sqrt{f})$. To see how this can be improved we recall that the relevant term comes from expressions (4.55) and (4.69) in [10]. In particular, now the precise estimate we need in order to obtain our claim can be written as

$$\left(\frac{1}{\ell}\sum_{i,j=1}^{\ell}\nu\left[\sqrt{(N_i+N_j)\,\nu_{i,j}[f]\,\mathcal{E}_{i,j}(\sqrt{f},\sqrt{f})}\right]\right)^2 \leqslant C_K N\,\mathcal{E}_\nu(\sqrt{f},\sqrt{f})\,, \tag{3.44}$$

where, following [10], we write $\nu_{i,j}[f] = \nu[f\,|\,\mathcal{F}_{i,j}]$, with $\mathcal{F}_{i,j}$ denoting the σ–algebra generated by $\{\eta_x, x \in (B_i \cup B_j)^c\}$. Here $\mathcal{E}_{i,j}$ stands for the Dirichlet form

$$\mathcal{E}_{i,j}(\sqrt{f},\sqrt{f}) = \sum_{x,y\in B_i\cup B_j}\nu_{i,j}[c_x(\nabla_{xy}\sqrt{f})^2]\,.$$

To prove (3.44) we observe that by Schwarz inequality for the combined measure $\frac{1}{\ell^2}\sum_{i,j}\nu[\cdot]$, the L.H.S. of (3.44) is bounded by

$$\left(\frac{1}{\ell}\sum_{i,j=1}^{\ell}\nu\Big[(N_i+N_j)\,\nu_{i,j}[f]\Big]\right)\left(\frac{1}{\ell}\sum_{i,j=1}^{\ell}\nu\left[\mathcal{E}_{i,j}(\sqrt{f},\sqrt{f})\right]\right)$$

The first term above is handled by observing that $\nu\left[(N_i+N_j)\nu_{i,j}[f]\right] = \nu\left[(N_i+N_j)f\right]$ and $\sum_i^\ell \nu\left[N_i f\right] = N$, since we are assuming $\nu[f] = 1$. Therefore (3.44) follows from the following estimate, which is easily verified

$$\frac{1}{\ell}\sum_{i,j=1}^{\ell}\nu\left[\mathcal{E}_{i,j}(\sqrt{f},\sqrt{f})\right] \leqslant C_K\mathcal{E}_\nu(\sqrt{f},\sqrt{f})\,.$$

This ends the proof of (3.40) assuming the result of Lemma 3.4.

Proof of Lemma 3.4. As in the proof of Lemma 3.3 we need to consider two regimes for the values of t.

Case $\frac{M}{K\sqrt{\rho}} \geqslant t \geqslant \sqrt{\frac{K}{N}}$. We use the following bound derived in [10], see (4.33) there:

$$\frac{1}{t} \log \nu\left[\exp t\Psi\right] \leqslant \frac{C}{t} + C\sqrt{N} + \frac{C_M N}{K}\, t\,, \qquad 0 \leqslant t \leqslant \frac{M}{K\sqrt{\rho}}\,. \tag{3.45}$$

If $t \geqslant \sqrt{\frac{K}{N}}$ we have $1/t \leqslant \frac{N}{K}\, t$ and $\sqrt{N} \leqslant \frac{N}{\sqrt{K}}\, t$, therefore (3.41) is contained in (3.45) in this case.

Case $t \leqslant \frac{M}{K\sqrt{\rho}} \wedge \sqrt{\frac{K}{N}}$. In this case we use the same strategy as in Lemma 3.3, in the case of small t. The function Ψ replaces now the function Φ, and the functions $K\bar{G}(\rho_j)$ play here the role of the functions $\bar{\varphi}_x$ defined in (3.27). We only sketch the arguments required to prove the needed estimates since they are essentially the same as in the case of small density. As in that case we may reduce the proof to suitable bounds on the expressions

$$E_1 := \frac{\ell}{2}\, \mu_{K,\rho}\left[(K\bar{G}(\rho_1))^2 e^{2tK\bar{G}(\rho_1)}\right] \mu_{K,\rho}\left[e^{2tK\bar{G}(\rho_1)}\right]^{\frac{\ell}{2}-1}\,,$$

$$E_2 := \frac{\ell}{2}\left(\frac{\ell}{2} - 1\right)\, \mu_{K,\rho}\left[K\bar{G}(\rho_1) e^{2tK\bar{G}(\rho_1)}\right]^2 \mu_{K,\rho}\left[e^{2tK\bar{G}(\rho_1)}\right]^{\frac{\ell}{2}-2}\,.$$

We recall that (see e.g. Corollary 6.4 in [19])

$$|\bar{G}(\rho_1) - G(\rho_1)| = |\nu[c_x] - \mu_{x,\rho}[c_x]| \leqslant C\frac{\sqrt{1+\rho}}{L}\,. \tag{3.46}$$

Therefore, using $\rho \geqslant \rho_0$

$$\mu_{K,\rho}\left[e^{2tK\bar{G}(\rho_1)}\right] \leqslant e^{CK\sqrt{\rho}t/L}\, \mu_{K,\rho}\left[e^{2tKG(\rho_1)}\right]\,. \tag{3.47}$$

Moreover as in (3.32)

$$\mu_{K,\rho}\left[e^{2tKG(\rho_1)}\right] \leqslant \exp\left\{2t^2 \mu_{K,\rho}\left[(KG(\rho_1))^2 e^{2tK|G(\rho_1)|}\right]\right\}\,. \tag{3.48}$$

An adaptation of estimates (4.12), (4.19) and (4.27) in [10] yields the following crucial bound:

$$\mu_{K,\rho}\left[(KG(\rho_1))^2 e^{2tK|G(\rho_1)|}\right] \leqslant C_M\, \rho\,, \qquad t \leqslant \frac{M}{K\sqrt{\rho}}\,. \tag{3.49}$$

Since $t \leqslant \sqrt{\frac{K}{N}}$, (3.47), (3.48) and (3.49) give

$$\mu_{K,\rho}\left[e^{2tK\bar{G}(\rho_1)}\right]^{\frac{\ell}{2}-1} \leqslant C_M\,. \tag{3.50}$$

Using again (3.46) we see that (3.47) and (3.49) imply

$$\mu_{K,\rho}\left[(K\bar{G}(\rho_1))^2 e^{2tK\bar{G}(\rho_1)}\right] \leqslant C_M\, \rho\,. \tag{3.51}$$

Summarizing, we have obtained $E_1 \leqslant C_M\, \ell\, \rho = C_M\, N/K$. The estimate on E_2 can be done in the same way as we did for the case of small density. In particular, using (3.51) we obtain $E_2 \leqslant C_M \ell^2 (\,K^2\, \rho/L^2 + t^2\, \rho^2)$. Since $t^2 \leqslant K/N$ this gives $E_2 \leqslant C_M\, N/K$. Therefore

$$\frac{1}{t} \log \nu\left[\exp t|\Psi|\right] \leqslant \frac{C_M N}{K}\, t\,, \qquad 0 \leqslant t \leqslant \frac{M}{K\sqrt{\rho}} \wedge \sqrt{\frac{K}{N}}\,.$$

This ends the proof of Lemma 3.4. ∎

Acknowledgments. We thank Paolo Dai Pra for several useful comments and discussions. We acknowledge the support of M.I.U.R.(Cofin).

References

[1] S.G. Bobkov, M. Ledoux, *On modified logarithmic Sobolev inequalities for Bernoulli and Poisson measures*, J. Funct. Anal. **156**, 347-365, 1998

[2] S.G. Bobkov, P. Tetali, *Modified logarithmic Sobolev inequalities in discrete settings*, preprint 2003, Preliminary version appeared in Proc. of the ACM STOC 2003, pages 287-296.

[3] N. Cancrini, F. Martinelli, *On the spectral gap of Kawasaki dynamics under a mixing condition revisited.* J. Math. Phys. **41**, no. 3, 1391-1423, 2000.

[4] N. Cancrini, F. Martinelli, C. Roberto, *The logarithmic Sobolev constant of Kawasaki dynamics under a mixing condition revisited*, Ann. Inst. H. Poincaré Probab. Statist. **38**, no. 4, 385–436, 2002.

[5] P. Caputo, *Spectral gap inequalities in product spaces with conservation laws*, in: Advanced Studies in Pure Mathematics, H. Osada and T. Funaki eds., Japan 2004.

[6] P. Caputo, P. Dai Pra, G. Posta, in preparation

[7] E. Carlen, M.C. Carvalho, M. Loss, *Determination of the spectral gap in Kac's master equation and related stochastic evolutions.* Acta Math. **191**, 1–54, 2003.

[8] D. Chafai, *Glauber versus Kawasaki for spectral gap and logarithmic Sobolev inequalities of some unbounded conservative spin systems*, Markov Process. Related Fields **9**, 341–362, 2003.

[9] P. Dai Pra, G. Posta, *Logarithmic Sobolev inequality for zero–range dynamics: independence of the number of particles*, Electron. J. Probab. **10**, 525–576, 2005

[10] P. Dai Pra, G. Posta, *Logarithmic Sobolev inequality for zero–range dynamics*, Ann. Probab. **33**, 2355–2401, 2005

[11] P. Dai Pra, A.M. Paganoni, G. Posta, *Entropy inequalities for unbounded spin systems.* Ann. Probab. **30**, 1959–1976, 2002

[12] P. Diaconis, L. Saloff–Coste, *Logarithmic Sobolev inequalities for finite Markov chains*, Ann. Appl. Probab. **6**, 695–750, 1996

[13] F. Gao, J. Quastel, *Exponential decay of entropy in the random transposition and Bernoulli–Laplace models.* Ann. Appl. Probab. **13**, 1591–1600, 2003

[14] S. Goel, *Modified logarithmic Sobolev inequalities for some models of random walk.* Stoc. Proc. Appl. **114**, 51–79, 2004

[15] E. Janvresse, C. Landim, J. Quastel, H.–T. Yau, *Relaxation to equilibrium of conservative dynamics. I. Zero-range processes*, Ann. Probab. **27**, 325–360, 1999

[16] C. Kipnis, C. Landim, *Scaling limits of interacting particle systems.* Grundlehren der Mathematischen Wissenschaften, 320. Springer-Verlag, Berlin, 1999

[17] C. Landim, C. N. Neto, *Poincaré and logarithmic Sobolev inequality for Ginzburg-landau processes in random environment*, Probability Theory and Related Fields **131**, 229-260, 2005

[18] C. Landim, G. Panizo, H. T. Yau, *Spectral gap and logarithmic Sobolev inequality for unbounded conservative spin systems*, Ann. Inst. H. Poincaré **38**, 739–777, 2002

[19] C. Landim, S. Sethuraman, S. R. S. Varadhan, *Spectral gap for zero-range dynamics.* Ann. Probab. **24**, 1871–1902, 1996

[20] M. Ledoux, *Logarithmic Sobolev inequalities for unbounded spin systems revisited.* Séminaire de Probabilités, XXXV, 167–194, Lecture Notes in Math., 1755, Springer, Berlin, 2001

[21] T. Y. Lee, H. T. Yau, *Logarithmic Sobolev inequality for some models of random walks.* Ann. Probab. **26** no. 4, 1855–1873, 1998

[22] S. T. Lu, H. T. Yau, *Spectral gap and logarithmic Sobolev inequality for Kawasaki and Glauber dynamics.* Comm. Math. Phys. **156**, 399-433, 1993

[23] F. Martinelli, *Relaxation times of Markov chains in statistical mechanics and combinatorial structures.* Encyclopedia of Mathematical Sciences Vol. **110**, Springer 2003

[24] Morris, B., *Spectral gap for the zero–range process with constant rate*, preprint `math.PR/0405161`

[25] C. Villani, *Cercignani's conjecture is sometimes true and always almost true.* Comm. Math. Phys. **234**, 455–490, 2003

[26] H. T. Yau, *Logarithmic Sobolev inequalities for generalized exclusion processes.* Probab. Theory Related Fields **109** no. 4, 507–538, 1997, .

Dip. Matematica, Universita' di Roma Tre, L.go S. Murialdo 1, 00146 Roma, Italy
E-mail address: caputo@mat.uniroma3.it

Dip. Matematica, Politecnico di Milano, P.za Leonardo da Vinci 32, I-20133 Milano, Italy
E-mail address: gustavo.posta@polimi.it